The Open University

MST121/Specimen and Solutions

Course Examination

Using Mathematics

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part I you should attempt as many questions as you can, writing your answers *in the spaces provided* inside this examination paper. Credit will be given for answers to no more than TWO questions in Part II. Your answers to this part should be written in the answer book provided.

In most questions, some marks will be awarded for intermediate steps in the working. A correct answer, not supported by working – as, for example, one taken from a calculator program – may not receive full credit.

80% of the available marks are assigned to Part I, and 20% to Part II. In the examiners' opinion, most candidates would make best use of their time by finishing as much as they can of Part I before starting Part II.

You are advised not to cross through any work until you have replaced it with another solution to the same question. In Part II of the paper, if you submit attempts to more than two questions, your best two scores will count towards your result.

Graph paper is available from the invigilator, if you feel that it would assist you in answering questions.

At the end of the examination

Check that you have completed the grid below, and have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

Examination No.							
Personal Identifier							

Put all your used answer books behind your question paper and put your signed desk record on top. Fix them all together with the paper fastener provided.

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PART I

Instructions

(i) *You should attempt as many questions as you can in this part of the examination.*

(ii) *Part I carries 80% of the available examination marks. Each question carries an indication of the number of marks that are allocated to it.*

(iii) *You should, as far as possible,* **record your answer to each question in this part in the space provided on the question paper.** *You are strongly advised to show all your working, including any rough working. If you need extra space, then you may continue your working in a separate answer book. If you do this, make sure that your work is clearly labelled.*

Question 1 – 3 marks

This question is about the sequence with closed form

$$x_n = 2 \times (0.4)^n \quad (n = 1, 2, 3, \ldots).$$

(a) Write down the first three terms of the sequence in the table below. [*1*]

x_1	
x_2	
x_3	

(b) Write down a recurrence system that generates the sequence. [*2*]

Question 2 – 4 marks

This question is about the sequence generated by the recurrence system

$$x_1 = 0.1, \quad x_{n+1} = 2x_n - 5 \quad (n = 1, 2, 3, \ldots).$$

(a) Write down the first four terms of the sequence in the table below. [*1*]

x_1	
x_2	
x_3	
x_4	

(b) Specify whether the sequence is arithmetic, geometric, or neither. [*1*]

(c) Write down a closed form for the sequence. [*2*]

Question 3 – 6 marks

A line in the (x, y)-plane has parametric equations $x = 2t - 3$, $y = 5t + 1$, where $t \geq 0$.

(a) Give the coordinates of two points on the line. [2]

(b) Eliminate t from the parametric representation to produce an equation for the line in terms of x and y. [2]

(c) Sketch the line. (A suitable range of values for t would be $0 \leq t \leq 5$.) [2]

Question 4 – 7 marks

The equation

$$x^2 + y^2 + 4x - 3y - \tfrac{11}{4} = 0$$

represents a circle.

(a) Express this equation in completed-square form, and hence find the coordinates of the centre and the radius of the circle. [3]

(b) Find the coordinates of the points where the circle cuts the y-axis. [2]

(c) Draw a sketch showing the circle in the plane. [2]

Question 5 – 4 marks

Find the value of $\sum_{i=6}^{23}(3i-4)$. [4]

Question 6 – 6 marks

For each sequence below, given by a closed-form formula, use reasoning to decide whether or not it converges. If it does converge, give the limit of the sequence.

(a) $u_n = n^2 + \dfrac{2}{n} \quad (n = 1, 2, 3, \ldots)$ [2]

(b) $v_n = \dfrac{10}{5 + 2^{-n}} \quad (n = 1, 2, 3, \ldots)$ [4]

Question 7 – 6 marks

(a) For the matrices **C** and **D** below, calculate **C** + **D** and **CD**.

$$\mathbf{C} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \qquad \mathbf{D} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$ [2]

(b) (i) Rewrite the following pair of simultaneous equations in matrix form.

$$2x + y = 1$$
$$5x + 3y = -2$$

(ii) Use matrices to solve this pair of equations. [4]

Question 8 – 4 marks

(a) Find the magnitude and direction of each of the following vectors, giving your answers correct to one decimal place.

$$\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} \qquad \mathbf{b} = 7\mathbf{i} - \mathbf{j}$$ [2]

(b) Find the magnitude and direction, to one decimal place, of the vector

$$\mathbf{c} = \mathbf{a} - \mathbf{b}.$$ [2]

Question 9 – 7 marks

(a) Show that the function

$$f(x) = \tfrac{1}{4}x^3 - 3x + 1$$

has stationary points at $x = \pm 2$. [2]

(b) Using the Second Derivative Test, classify each of these stationary points as a local maximum or a local minimum. [2]

(c) Draw a sketch of the graph of the function f, indicating the coordinates of the stationary points and the y-intercept. [3]

Question 10 – 6 marks

(a) Differentiate the function

$$f(x) = \frac{5}{x^2} + 7\sin(2x),$$

identifying any general results that you use. [3]

(b) Find the indefinite integral of

$$g(t) = -4e^{9t} + \frac{6}{t} \quad (t > 0),$$

identifying any general results that you use. [3]

Question 11 – 4 marks

The equation

$$h = 32t - 5t^2$$

is to be used as a model for the motion of a ball projected vertically upwards with a particular initial speed. Here h is the height (in metres) above the point of projection, and t is the time (in seconds) since the moment of projection.

(a) (i) Express the velocity v of the ball in terms of t.

(ii) What is the speed with which the ball is projected? [2]

(b) What is the maximum height of the ball above the point of projection? [2]

Question 12 – 3 marks

Find, in explicit form, the general solution of the differential equation

$$\frac{dy}{dx} = e^{-y} \sin x.$$

[3]

Question 13 – 6 marks

Two coins are tossed together repeatedly.

(a) Write down the probability that in a single toss, both coins land heads up (a double head). [*1*]

(b) Find the probability that none of the first five tosses results in a double head. [*2*]

(c) Find the probability that there is at least one double head in the first five tosses. [*1*]

(d) If the two coins are tossed until a double head is obtained, how many times would you expect them to be tossed? (That is, what is the mean number of tosses of the two coins required for a double head to occur?) [*2*]

Question 14 – 4 marks

The variation in the heights of eight-year-old boys may be modelled by a normal distribution with mean 126 cm and standard deviation 5.4 cm.

(a) Write down the mean and the standard deviation of the sampling distribution of the mean for samples of size 36 from this population. [*2*]

(b) Calculate a range of values within which the mean height of approximately 95% of samples of 36 eight-year-old boys will lie. [*2*]

Question 15 – 7 marks

The wing lengths of 25 male and 31 female reed buntings are measured, and the sample mean and sample standard deviation are calculated separately for the males and for the females. The statistics are given in the table below; the units are millimetres.

	Males	Females
Sample size	25	31
Mean	81.1	78.2
Standard deviation	2.28	2.76

It is proposed to use the two-sample z-test to investigate whether there is a difference between the mean wing length of male reed buntings and the mean wing length of female reed buntings.

(a) Use symbols to write down appropriate null and alternative hypotheses for the test. Explain the meanings of any symbols that you use (other than H_0 and H_1). [2]

(b) Calculate the test statistic. [2]

(c) What conclusions can you draw about the wing lengths of male and female reed buntings? [3]

Question 16 – 3 marks

In a study of physical measurements within families, data are collected for a large number of families on the heights in centimetres of husbands and their wives. The data on the heights of the men (x) and the heights of their wives (y) are used to calculate a least squares fit line, whose equation is $y = 0.95x - 2.2$.

(a) According to the model described by the least squares fit line, what is the mean height of wives of men who are 180 cm tall? [*1*]

(b) One of the men in the sample is 180 cm tall, and his wife is 167 cm tall. Use the least squares fit line to calculate the residual for this couple. [*1*]

(c) Suppose that you wish to fit a straight-line model which could be used to estimate the mean height of husbands for wives of various heights. How would you do this? [*1*]

PART II

Instructions

(i) *Credit will be given for answers to no more than* **TWO** *questions from this part of the examination.*

(ii) *Each question in this part carries 10% of the total marks for the examination.*

(iii) *You may answer the questions in any order. Write your answers in the answer book(s) provided, beginning each question on a new page.*

(iv) *Show all your working.*

Question 17

Consider the function $f(x) = 2 + 14x - 2x^2$.

(a) Calculate the images $f(0)$ and $f(2)$. [*1*]

(b) The graph of the function f is a parabola. Express $f(x)$ in completed-square form, and hence find the coordinates of the vertex of the parabola. [*4*]

(c) Explain how the graph of $y = f(x)$ may be obtained from that of $y = x^2$ by appropriate translations and/or scalings. [*3*]

(d) Sketch the graph of $y = f(x)$. [*2*]

Question 18

A population of birds may be divided into juveniles (up to one year old), immatures (between one and four years old) and adults (over four years old). A model of the population is constructed on the following assumptions about the changes in any given year.

- The number of birds born during the year, and surviving to the end of the year, is 80% of the adult population at the end of the previous year.
- 80% of juveniles from the end of the previous year survive as immatures.
- 60% of immatures from the end of the previous year survive as immatures, and 30% of them survive as adults.
- 90% of adults from the end of the previous year survive.

Let J_n, I_n and A_n be the numbers of juveniles, immatures and adults at the end of year n.

(a) Express each of J_{n+1}, I_{n+1} and A_{n+1} in terms of the previous values J_n, I_n and A_n. [*4*]

(b) Let $\mathbf{P}_n$ be the vector

$$\mathbf{P}_n = \begin{pmatrix} J_n \\ I_n \\ A_n \end{pmatrix}.$$

Write down the corresponding matrix model of the population in the form $\mathbf{P}_{n+1} = \mathbf{M}\mathbf{P}_n$, where $\mathbf{M}$ is a 3×3 matrix. [2]

(c) According to Mathcad, the matrix

$$\mathbf{B} = \begin{pmatrix} 2.813 & 1.25 & -2.5 \\ -3.75 & 0 & 3.333 \\ 1.25 & 0 & 0 \end{pmatrix}$$

is the inverse of the matrix $\mathbf{M}$ in part (b).

At the end of 2002, the estimated numbers of juveniles, immatures and adults were, respectively, 600, 950 and 900. Calculate the estimated number of birds in each category at the end of 2001. [*4*]

Question 19

This question concerns the function $f(x) = xe^{-x/3}$. Using the Product Rule, it can be shown that $f'(x) = (1 - \frac{1}{3}x)e^{-x/3}$.

(You are *not* expected to show this.)

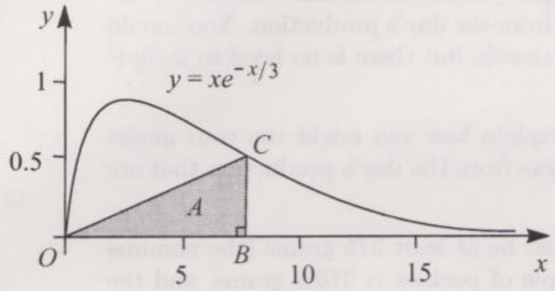
(a) Use the given derivative, together with some algebraic manipulation, to show that

$$\int xe^{-x/3}\,dx = -3(x+3)e^{-x/3} + c,$$

where c is an arbitrary constant. [3]

(b) Find the area under the graph of $f(x)$ from $x = 0$ to $x = 3$, giving your answer to 3 decimal places. [2]

The remainder of this question relates to the triangle OBC, shown shaded in the figure below, for which B has coordinates $(x, 0)$ and C lies vertically above B on the graph of $f(x)$.



(c) Write down an expression, in terms of x, for the area A of the triangle OBC. [*1*]

(d) Find the value of x for which A is a maximum, and calculate this maximum area to 3 decimal places. [*4*]

(There is no need to justify that the value found is indeed a maximum.)

Question 20

A packer takes a sample of 60 packets of cereal with nominal weight 375 grams from a day's production, and weighs the contents of each packet. The weights are summarised in the table below.

Weight in grams	Frequency
355–360	1
360–365	1
365–370	3
370–375	19
375–380	22
380–385	12
385–390	2

(a) Draw a frequency diagram to represent these data. Is the distribution of weights in the sample symmetrical, left-skewed or right-skewed? Briefly explain your answer. *[3]*

(b) (i) Sketch a curve which you think might provide a reasonable model for the variation in the weights of packets from the day's production. You should include a horizontal scale on your sketch, but there is no need to include a vertical scale.

(ii) With the assistance of a sketch, explain how you might use your model to estimate the proportion of packets from the day's production that are more than 8 grams underweight. *[4]*

(c) By law, the mean weight of packets must be at least 375 grams (the nominal weight). The mean weight of the sample of packets is 376.2 grams, and the sample standard deviation is 5.51 grams.

(i) Calculate a 95% confidence interval for the mean weight of all packets from the day's production.

(ii) What can the packer conclude about the mean weight of packets from the day's production? *[3]*

[END OF QUESTION PAPER]

Solutions

PART I

Question 1

(a)

x_1	0.8
x_2	0.32
x_3	0.128

[1]

(b) $x_1 = 0.8, \quad x_{n+1} = 0.4x_n \quad (n = 1, 2, 3, \ldots).$ [2]

Question 2

(a)

x_1	0.1
x_2	−4.8
x_3	−14.6
x_4	−34.2

[1]

(b) This is neither an arithmetic nor a geometric sequence (although it is linear). [1]

(c) This is a linear recurrence sequence. A closed form is

$$x_n = -4.9 \times 2^{n-1} + 5 \quad (n = 1, 2, 3, \ldots).$$ [2]

(Here, in terms of the standard recurrence sequence parameters, $a = 0.1$, $d = -5$, $r = 2$.)

Question 3

(a) $t = 0$ gives $(-3, 1)$, [1]
$t = 1$ gives $(-1, 6)$. [1]
(There are also many others.)

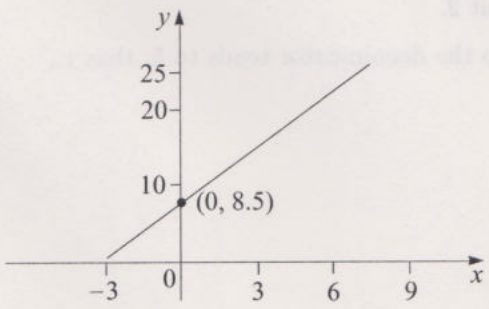
(b) $x = 2t - 3$ gives $t = \dfrac{x+3}{2}$, [1]

so $y = 5\left(\dfrac{x+3}{2}\right) + 1$; [$\frac{1}{2}$]

that is,

$$2y = 5x + 17, \quad (x \geq -3).$$ [$\frac{1}{2}$]

(c)



[*1 gradient*]
[*1 intercept*]

Question 4

(a) Combining the terms in x and in y, we obtain

$$(x+2)^2 + (y - \tfrac{3}{2})^2 = 4 + \tfrac{9}{4} + \tfrac{11}{4} = 9. \qquad [1\tfrac{1}{2}]$$

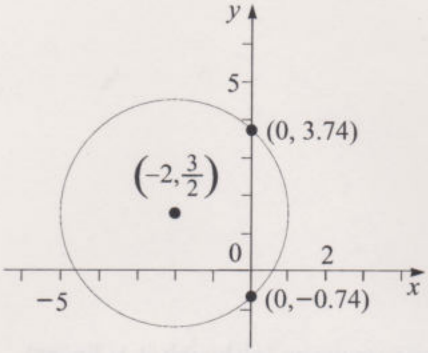
So the centre is $(-2, \frac{3}{2})$, and the radius is 3. $[1\frac{1}{2}]$

(b) The circle cuts the y-axis where $x = 0$, i.e. where $y^2 - 3y - \frac{11}{4} = 0$, $[1\frac{1}{2}]$

giving $y = \frac{1}{2}(3 \pm \sqrt{20}) = \frac{3}{2} \pm \sqrt{5}$, that is, $y = 3.74$ and $y = -0.74$ (to 2 d.p.).

It cuts the y-axis at the points $(0, \frac{3}{2} + \sqrt{5})$ and $(0, \frac{3}{2} - \sqrt{5})$, that is, at $(0, 3.74)$ and $(0, -0.74)$. $[\frac{1}{2}]$

(c)



[2]

Question 5

$$\sum_{i=6}^{23}(3i-4) = 3\sum_{i=6}^{23} i - 18 \times 4 \qquad [1]$$

$$= 3\left(\sum_{i=1}^{23} i - \sum_{i=1}^{5} i\right) - 72$$

$$= 3(\tfrac{1}{2} \times 23 \times 24 - \tfrac{1}{2} \times 5 \times 6) - 72 \qquad [2]$$

$$= 3 \times 261 - 72$$

$$= 711. \qquad [1]$$

Question 6

(a) This sequence does not converge. The term n^2 becomes arbitrarily large as n increases, while $\dfrac{2}{n}$ tends to 0. Consequently, u_n increases without bound and does not have a limit. [2]

(b) This sequence does converge, to the limit 2. [2]

For large n, $2^{-n} = (\frac{1}{2})^n$ is close to 0, so the denominator tends to 5; thus v_n tends to $\frac{10}{5} = 2$. [2]

Question 7

(a) $\mathbf{C}+\mathbf{D}=\begin{pmatrix}5&0\\0&5\end{pmatrix}$ *[1]*

$\mathbf{CD}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ *[1]*

(b) (i) The matrix form of the equations is

$$\begin{pmatrix}2&1\\5&3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1\\-2\end{pmatrix}.$$ *[1]*

(ii) The inverse matrix of $\mathbf{C}=\begin{pmatrix}2&1\\5&3\end{pmatrix}$ is

$$\mathbf{C}^{-1}=\frac{1}{1}\begin{pmatrix}3&-1\\-5&2\end{pmatrix}=\begin{pmatrix}3&-1\\-5&2\end{pmatrix}.$$ *[1]*

So

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}3&-1\\-5&2\end{pmatrix}\begin{pmatrix}1\\-2\end{pmatrix}$$

$$=\begin{pmatrix}3+2\\-5-4\end{pmatrix}=\begin{pmatrix}5\\-9\end{pmatrix}.$$ *[1]*

Hence the solution is

$x=5,\ y=-9.$ *[1]*

Alternatively, from part (a), as $\mathbf{CD}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ then $\mathbf{D}$ is the inverse of $\mathbf{C}$ and so $\begin{pmatrix}x\\y\end{pmatrix}=\mathbf{D}\begin{pmatrix}1\\-2\end{pmatrix}$; then continue as above.

Question 8

(a) $|\mathbf{a}|=\sqrt{2^2+2^2}=\sqrt{8}=2.8$ to one decimal place. Since the components are $a_1=a_2=2$, the direction is $\theta=45°$. *[1]*

$|\mathbf{b}|=\sqrt{7^2+(-1)^2}=\sqrt{50}=7.1$ to one decimal place. Since the components are $b_1=7$, $b_2=-1$, we have

$$\phi=\arctan\left|\frac{-1}{7}\right|=\arctan\tfrac{1}{7}\simeq 8.1°.$$ *[1]*

Also, $(7,-1)$ lies in the fourth quadrant, so the direction is $\theta\simeq -8.1°$.

(b) We have $\mathbf{c}=\mathbf{a}-\mathbf{b}=2\mathbf{i}+2\mathbf{j}-(7\mathbf{i}-\mathbf{j})=-5\mathbf{i}+3\mathbf{j}$. *[1]*

So $|\mathbf{c}|=\sqrt{(-5)^2+3^2}=\sqrt{34}=5.8$ to one decimal place. Since the components are $c_1=-5$, $c_2=3$, we have

$$\phi=\arctan\left|\frac{3}{-5}\right|=\arctan\tfrac{3}{5}\simeq 31.0°.$$ *[1]*

Also, $(-5,3)$ lies in the second quadrant, so the direction is $\theta\simeq 149.0°$.

Question 9

(a) The derivative of $f(x) = \frac{1}{4}x^3 - 3x + 1$ is

$$f'(x) = \tfrac{3}{4}x^2 - 3 = \tfrac{3}{4}(x-2)(x+2), \qquad [1]$$

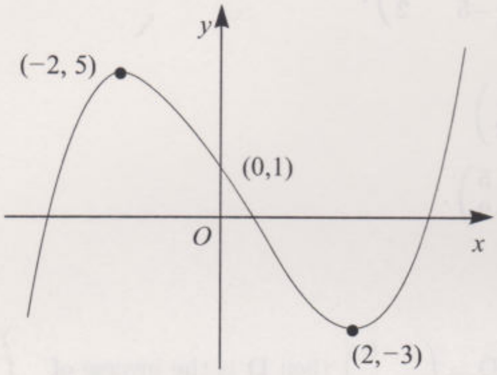
so the stationary points, corresponding to $f'(x) = 0$, are at $x = \pm 2$. [1]

(b) The second derivative is $f''(x) = \frac{3}{2}x$. [$\frac{1}{2}$]

Since $f''(-2) = -3 < 0$, there is a local maximum at $x = -2$. [1]

Since $f''(2) = 3 > 0$, there is a local minimum at $x = 2$. [$\frac{1}{2}$]

(c) Since $f(-2) = 5$ and $f(2) = -3$, the stationary points are at $(-2, 5)$ and $(2, -3)$. The y-intercept is $f(0) = 1$.



[$1\frac{1}{2}$ general shape]
[1 stationary points]
[$\frac{1}{2}$ intercept]

Question 10

(a) Differentiating $5/x^2 = 5x^{-2}$, using the Constant Multiple Rule, gives $5(-2x^{-3}) = -10/x^3$. [1]

Differentiating $7\sin(2x)$, using the Constant Multiple Rule, gives $7(2\cos(2x)) = 14\cos(2x)$. [1]

Combining these results, using the Sum Rule for differentiation, gives

$$f'(x) = -\frac{10}{x^3} + 14\cos(2x). \qquad [1]$$

(b) Using the Constant Multiple Rule gives

$$\int(-4e^{9t})\,dt = -4\int e^{9t}\,dt = -4 \times \tfrac{1}{9}e^{9t} = -\tfrac{4}{9}e^{9t}. \qquad [1]$$

Since $t > 0$, using the Constant Multiple Rule gives

$$\int \frac{6}{t}\,dt = 6\int \frac{1}{t}\,dt = 6\ln t. \qquad [1]$$

Combining these results, using the Sum Rule for integration, gives

$$\int g(t)\,dt = -\tfrac{4}{9}e^{9t} + 6\ln t + c \quad (t > 0). \qquad [1]$$

Question 11

(a) (i) The velocity v is given by $v = dh/dt = 32 - 10t$. [1]

(ii) The speed with which the ball is projected is the speed when $t = 0$, i.e. $32\,\mathrm{m\,s^{-1}}$. [1]

(b) The ball is at its maximum height when $v = 32 - 10t = 0$, i.e. when $t = 3.2\,\mathrm{s}$. So the maximum height is $32(3.2) - 5(3.2)^2 = 51.2\,\mathrm{m}$. [2]

Question 12

Separating the variables, we obtain

$$\int e^y\,dy = \int \sin x\,dx. \qquad [1]$$

On performing the integrations, this gives

$$e^y = c - \cos x, \qquad [1\tfrac{1}{2}]$$

and so, making y the subject, the general solution in explicit form is

$$y = \ln(c - \cos x). \qquad [\tfrac{1}{2}]$$

Question 13

(a) Either count outcomes or use the multiplication rule:

$$P(\text{double head}) = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}. \qquad [1]$$

(b) The probability that one toss of the two coins does *not* result in a double head is $1 - \frac{1}{4} = \frac{3}{4}$. [1]

The probability of no double heads in five tosses is $\left(\frac{3}{4}\right)^5 \simeq 0.2373$. [1]

(c) The probability of at least one double head in five tosses is $1 - 0.2373 = 0.7627$. [1]

(d) The number of tosses required to obtain a double head has a geometric distribution with parameter $p = \frac{1}{4}$. [1]

So the mean number of tosses required to obtain a double head is

$$\frac{1}{p} = \frac{1}{1/4} = 4. \qquad [1]$$

Question 14

(a) The sampling distribution of the mean for samples of size 36 has

mean 126 cm, and [1]

standard deviation $\dfrac{5.4}{\sqrt{36}} = 0.9\,\text{cm}$. [1]

(b) Approximately 95% of sample means will lie within 1.96 standard deviations of the mean; that is, between

$$126 - (1.96 \times 0.9) \simeq 124.2\,\text{cm}$$

and

$$126 + (1.96 \times 0.9) \simeq 127.8\,\text{cm}. \qquad [2]$$

Question 15

(a) $H_0 : \mu_M = \mu_F$,
$H_1 : \mu_M \neq \mu_F$, [*1*]

where μ_M is the mean wing length of the population of male reed buntings, and μ_F is the mean wing length of the population of female reed buntings. [*1*]

(b) The estimated standard error of the difference between the two means is

$$ESE = \sqrt{\frac{s_M^2}{n_M} + \frac{s_F^2}{n_F}} = \sqrt{\frac{2.28^2}{25} + \frac{2.76^2}{31}} = 0.673\,546\,6\ldots. \qquad [1]$$

The test statistic is

$$z = \frac{\overline{x}_M - \overline{x}_F}{ESE} = \frac{81.1 - 78.2}{0.673\,546\,6\ldots} \simeq 4.31. \qquad [1]$$

(c) Since the test statistic is $z = 4.31 > 1.96$, we reject the null hypothesis at the 5% significance level in favour of the alternative hypothesis. [*1*]

We conclude that the mean wing length of male reed buntings is not equal to the mean wing length of female reed buntings. [*1*]

The sample mean is greater for the males than for the females, which suggests that the mean wing length of male reed buntings is greater than the mean wing length of female reed buntings. [*1*]

Question 16

(a) The predicted mean height of wives of men who are 180 cm tall is

$$(0.95 \times 180) - 2.2 = 168.8\,\text{cm}. \qquad [1]$$

(b) RESIDUAL = DATA − FIT = $167 - 168.8 = -1.8$ cm. [*1*]

(c) Take wife's height as the explanatory variable (x) and husband's height as the dependent variable (y), and calculate the least squares fit line from given data. [*1*]

PART II

Question 17

(a) $f(0) = 2$ and $f(2) = 22$. [1]

(b) We obtain

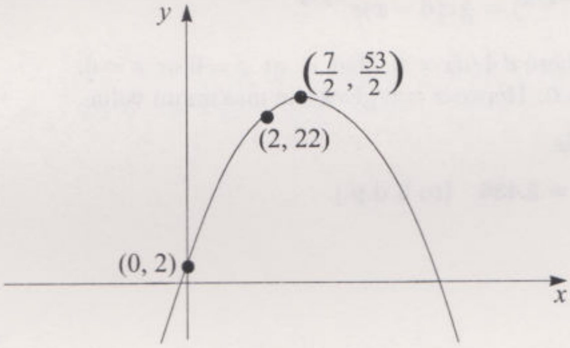
$$\begin{aligned} f(x) = 2 + 14x - 2x^2 &= -2\left(x^2 - 7x - 1\right) \\ &= -2\left(\left(x - \tfrac{7}{2}\right)^2 - \tfrac{49}{4} - 1\right) \\ &= -2\left(\left(x - \tfrac{7}{2}\right)^2 - \tfrac{53}{4}\right) \\ &= -2\left(x - \tfrac{7}{2}\right)^2 + \tfrac{53}{2}. \end{aligned}$$ [3]

Hence the vertex of the parabola is at $\left(\frac{7}{2}, \frac{53}{2}\right)$. [1]

(c) Perform first a y-scaling by -2;
then a translation by $\frac{7}{2}$ to the right;
then a translation through $\frac{53}{2}$ upwards. [3]

Alternative answers are possible, for example, the following:
Perform first a translation by $\frac{7}{2}$ to the right;
then a translation by $\frac{53}{4}$ downwards;
then a y-scaling by -2.

(d)



[2]

Question 18

(a) $J_{n+1} = 0.8A_n$, [1]
$I_{n+1} = 0.8J_n + 0.6I_n$, [2]
$A_{n+1} = 0.3I_n + 0.9A_n$. [1]

(b) The matrix model of the population is $\mathbf{P}_{n+1} = \mathbf{MP}_n$, that is,

$$\begin{pmatrix} J_{n+1} \\ I_{n+1} \\ A_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0.8 \\ 0.8 & 0.6 & 0 \\ 0 & 0.3 & 0.9 \end{pmatrix} \begin{pmatrix} J_n \\ I_n \\ A_n \end{pmatrix}.$$ [2]

(c) Using $\mathbf{P}_{2001} = \mathbf{BP}_{2002}$ with $\mathbf{P}_{2002} = \begin{pmatrix} 600 \\ 950 \\ 900 \end{pmatrix}$, we obtain $\mathbf{P}_{2001} = \begin{pmatrix} 625 \\ 750 \\ 750 \end{pmatrix}$. [3]

Hence, at the end of 2001 there are

625 juveniles, 750 immatures, 750 adults. [1]

Question 19

(a) It follows from the given derivative that

$$xe^{-x/3} = \int (1 - \tfrac{1}{3}x)e^{-x/3}\,dx$$
$$= \int e^{-x/3}\,dx - \tfrac{1}{3}\int xe^{-x/3}\,dx, \qquad [1]$$

by the Sum and Constant Multiple Rules for integrals. Hence we have

$$\int xe^{-x/3}\,dx = 3\left(\int e^{-x/3}\,dx - xe^{-x/3}\right) \qquad [1]$$
$$= 3\left(-3e^{-x/3} - xe^{-x/3}\right) + c \qquad [1]$$
$$= -3(x+3)e^{-x/3} + c,$$

where c is an arbitrary constant.

(b) Using the result of part (a), the required area is

$$\int_0^3 xe^{-x/3}\,dx = \left[-3(x+3)e^{-x/3}\right]_0^3 \qquad [1]$$
$$= -18e^{-1} - (-9) = 9(1 - 2e^{-1}) = 2.378 \quad \text{(to 3 d.p.).} \qquad [1]$$

(c) The area of the triangle is $\frac{1}{2}$ × base × height, that is,

$$A = \tfrac{1}{2} \times x \times xe^{-x/3} = \tfrac{1}{2}x^2e^{-x/3}. \qquad [1]$$

(d) Using the Product Rule, we have

$$\frac{dA}{dx} = xe^{-x/3} \;+\; \tfrac{1}{2}x^2(-\tfrac{1}{3}e^{-x/3}) = \tfrac{1}{6}x(6-x)e^{-x/3}. \qquad [2]$$

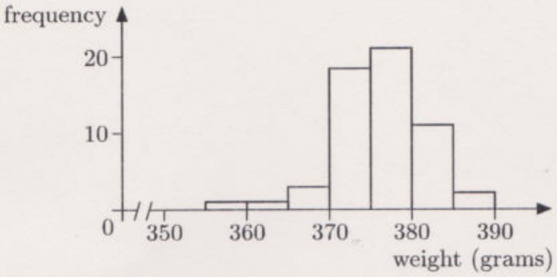
The maximum value of A occurs where $dA/dx = 0$, that is, at $x = 0$ or $x = 6$. However, $A(0) = 0$, whereas $A(6) > 0$. Hence $x = 6$ gives the maximum value. [1]

The corresponding maximum area is

$$A(6) = \tfrac{1}{2} \times 6^2e^{-2} = 18e^{-2} = 2.436 \quad \text{(to 3 d.p.).} \qquad [1]$$

Question 20

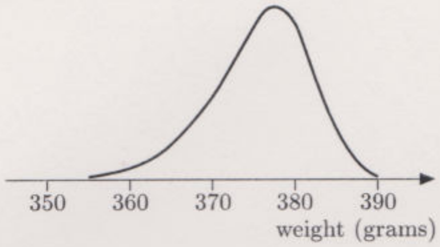
(a)



[2]

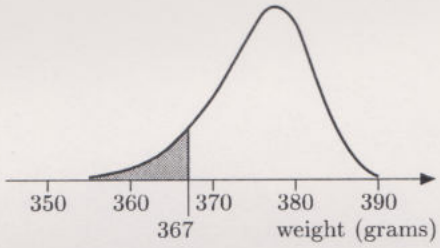
The distribution is left-skew (but not markedly so) since it has a longer left tail than right tail. [1]

(b) (i)



[1]

(ii)



[1]

Assuming that the total area under the curve in part (b)(i) is equal to 1, the shaded area above (to the left of 367 grams) represents the proportion of packets more than 8 grams underweight. So we should obtain an estimate for this area in order to find the proportion of packets more than 8 grams underweight. [2]

(c) (i) A 95% confidence interval for the mean weight of packets from the day's production is given by

$$\left(\overline{x} - 1.96\frac{s}{\sqrt{n}},\ \overline{x} + 1.96\frac{s}{\sqrt{n}}\right), \qquad [1]$$

that is,

$$\left(376.2 - \left(1.96 \times \frac{5.51}{\sqrt{60}}\right),\ 376.2 + \left(1.96 \times \frac{5.51}{\sqrt{60}}\right)\right)$$
$$= (374.8, 377.6). \qquad [1]$$

(ii) The 95% confidence interval extends below 375 grams, so the mean weight of the day's production may well be below the nominal weight. (It would be necessary for the packer to check some more packets.) [1]

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